Gravitational Dynamics of an Infinite Shuffled Lattice: Particle Coarse-grainings, Non-linear Clustering and the Continuum Limit

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Abstract

We study the evolution under their self-gravity of particles evolving from infinite "shuffled lattices" initial conditions. We focus here specifically on the comparison between the evolution of such a system and that of "daughter" coarse-grained particle distributions. These are sparser (i.e. lower density) particle distributions, defined by a simple coarse-graining procedure, which share the same large-scale mass fluctuations. We consider both the case that such coarse-grainings are performed (i) on the initial conditions, and (ii) at a finite time with a specific additional prescription. In numerical simulations we observe that, to a first approximation, these coarse-grainings represent well the evolution of the two-point correlation properties over a significant range of scales. We note, in particular, that the form of the two-point correlation function in the original system, when it is evolving in the asymptotic "self-similar" regime, may be reproduced well in a daughter coarse-grained system in which the dynamics are still dominated by two-body (nearest neighbor) interactions. This provides a simple physical description of the origin of the form of part of the asymptotic non-linear correlation function. Using analytical results on the early time evolution of these systems, however, we show that small observed differences between the evolved system and its coarse-grainings *at the initial time* will in fact diverge as the ratio of the coarse-graining scale to the original inter-particle distance increases. The second coarse-graining studied, performed at a finite time in a specified manner, circumvents this problem. It also makes more physically transparent why gravitational dynamics from these initial conditions tends toward a "self-similar" evolution. We finally discuss the precise definition of a limit in which a continuum (specifically Vlasov-like) description of the observed linear and non-linear evolution should be applicable. This requires the introduction of an additional intrinsic length scale (e.g. a physical smoothing in the force at small scales), which is kept fixed as the particle density diverges. In this limit the different coarse-grainings are equivalent and leave the evolution of the "mother" system invariant.

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I. INTRODUCTION

The dynamics of infinite self-gravitating systems of point particles starting from quasi-uniform initial conditions is a straightforward, well posed, but essentially unsolved problem of out of equilibrium statistical mechanics. While numerical simulation of such systems has developed impressively in scale and sophistication in the last few decades — mostly driven by the relevance of the problem in cosmology — analytic understanding of the evolution of the clustering observed remains very limited. The work reported in this article is a continuation of a study of this problem reported by us recently [1]. We study a simplified version of the problem posed in cosmology, considering pure Newtonian gravity in a static euclidean universe and a very simple class of "shuffled lattice" initial conditions (IC).

In [1] we have presented in detail the results of a numerical study of this system. We have found that, as soon as strong positive two-point correlations begin to develop, they are characterized by a simple spatio-temporal scaling relation, in which the strong spatial correlations at any time may be inferred from those at a precedent time by a simple rescaling of the spatial variables. Further the function of time specifying this dynamical scaling of the characteristic spatial scale tends, after a transient

period, to a form which can be determined solely from the linearized fluid equations describing the evolution of the small fluctuations at large scales. This asymptotic dynamical scaling behavior is completely analogous to that which has been observed in cosmological simulations, starting from more complicated IC (correlated perturbations of lattices) and in an expanding universe, and referred to in this latter context as "self-similarity". The amplitude of the two-point correlation function is also observed in this asymptotic regime to be essentially independent of the mean particle density over a large range of scale. This suggests that the full non-linear dynamics of these simulations, once this asymptotic scaling behavior is approached, might be well approximated solely within the framework of a continuous fluid-like description of the system. We have found, however, that the *form* of the two-point correlation function (i.e. its dependence on separation rather than its temporal scaling behavior) attained in this asymptotic scaling regime is already very well approximated, at least for modest amplitudes, by that at early times where the dynamics at these scales is explicitly particle-like in nature. Indeed the correlation function which develops at these times is very well approximated by assuming that the dynamics is that of particles interacting only with their few nearest neighbor particles. In other words, while the spatio-temporal scaling law seems to be determined by only the large scale fluid linear dynamics, the form of the scaling function looks to depend on the discrete aggregation dynamics between elementary objects. This suggests that the discreteness of the system may be an essential ingredient even to understand its long-time behavior.

In this paper we study further the dynamical evolution in this same class of models (i.e. with "shuffled lattice" IC in a static universe). Specifically we focus here on the comparison between the evolution of a given such IC and the evolution of other point distributions defined from them by a simple coarse-graining procedure: at a chosen time the mass in coarse-graining cells, defined by a lattice, is aggregated into a single mass; this new point mass is attributed the position and velocity of the center of mass of the mass in the cell. This is a procedure which evidently modifies the system up to a certain scale — that characteristic of the coarse-graining — but leaves the mass and velocity fluctuations at larger scales essentially unperturbed. The comparison of the original and coarse-grained (CG) system allows one to infer useful physical information about the dynamics. Specifically it allows one to study the role of fluctuations at different scales in the system, and the degree to which a discrete or continuous description of the dynamics can be appropriate. We will see that the study will allow us to understand better the physics underlying the asymptotic "self-similar" evolution, and also to address more precisely the question of the continuum limit of the system. These latter considerations are of interest, as we will explain in the conclusions, in the context of simulations in cosmology. Such simulations can be seen as particle coarse-grainings in the sense defined here of the "true" self-gravitating matter described by cosmological models[1]. A clear understanding of the continuum limit is important as it should correspond to the theoretical cosmological model. Discreteness effects in these simulations are the differences between the CG and this underlying continuum model.

In the introduction to [1] we have given a brief review of the relevant literature, on both self-gravitating systems and other long-range interacting systems. The notion of "particle coarse-graining", which is the central one in this paper, is simply a specific kind of coarse graining procedure introduced in statistical physics (see, e.g.[2]). It is evidently not unnatural to consider such schemes in approaching the dynamics of systems manifesting scaling behavior, as has been done widely in many different systems in condensed matter physics with well known results. In the context of self-gravitating systems various authors have studied self-gravitating dynamics applying such concepts (see, e.g., [3, 4, 5, 6, 7]). These works, however, study theoretical descriptions of these systems in a continuum limit. In this work we consider, instead, the fully discrete systems, numerically simulated, applying coarse-graining procedures which relate one such discrete system to another. Indeed one of the questions we address in this work is, as mentioned above, the validity of a continuum description of the dynamics. One of our findings is that the coarse-grained system, in a phase in which its dynamics are manifestly discrete, already traces well the behavior of the original system. The only previous work we are aware of which has actually applied a discrete "renormalization" procedure is that of [8, 9], which attempts to determine the mass distribution which results from power law initial conditions using such a scheme under the *assumption* that the evolution is self-similar. Numerical studies of discreteness effects in cosmological N body simulations (e.g. [10]) also implicitly consider the effect of different discrete coarse-grainings, albeit not defined as such in the simple way considered here. Indeed, as mentioned above, one of the goals of this work is to clarify problems in this context by a well controlled study of much simpler systems.

The paper is organized as follows. In the next section we summarize briefly the essential definitions and the results of the preceding paper [1]. We also summarize relevant results derived recently by the present authors and B. Marcos [11, 12] on a perturbative description of the early times dynamics of self-gravitating points perturbed off a lattice. We underline the slight differences in the systems studied here compared to that in [1] — a different probability distribution function for the initial "shuffling" and also a different ascription of initial velocities.

[1] For typical "cold dark matter" models there is a factor of the order of 10^{70} between the microscopic number density of the physical particles and the number density of the "macro-particles" used in the largest simulations performed up to now.

To explain this latter point we include a short discussion of the "Zeldovich approximation" to the evolution of a pressure-less self-gravitating fluid (This approximation is also important for the discussion in Sec. V of the paper). In Sec. III we define precisely and discuss the construction of the coarse-grained particle distributions from a given initial particle system. In the following section we report our study of the comparison between a given SL and a set of CG of the initial conditions, while in Sec. V we do the same for a set of CG defined at a given scale at finite time in the course of the evolution. In the final section we summarize our results and conclusions, and discuss what we have learned about the dynamics of these systems. Specifically we discuss the question of the appropriateness of a continuum description of the observed non-linear dynamics, and we indicate some paths for future research on this question.

II. BASIC DEFINITIONS AND BACKGROUND RESULTS

In this section we give the essential definitions and background for the paper. Firstly we summarize the basic definitions of the initial conditions studied and of the quantities used to characterize the systems. Except for a few minor points this discussion essentially summarizes an analogous one in [1], where the interested reader can find further details. One minor difference is that we consider here the analysis also of particle distributions in which the masses of all points are not equal. We then summarize the essential results of [1], and then also those of [11] which we also make use of in this paper. Finally, we discuss briefly the so-called "Zeldovich approximation" which is an important perturbative approximation to the evolution of self-gravitating systems in the fluid limit, and therefore an approximation to the evolution of the particle system in a certain regime (long wavelengths and small amplitude perturbations). We will use it specifically, as it is commonly done in cosmological simulations, in the choice of initial conditions for the particle velocities, and also at various points in our analysis.

A. Definition of a Shuffled Lattice

We call *shuffled lattice* (SL) the particle distribution obtained by applying a random displacement independently to each particle on a simple cubic lattice. Thus the position of a particle, initially on the lattice site $\boldsymbol{R}$, is given by $\boldsymbol{x}(\boldsymbol{R}) = \boldsymbol{R} + \boldsymbol{u}(\boldsymbol{R})$ where the vectors $\boldsymbol{u}(\boldsymbol{R})$ are random and specified by a probability density functional

$$\mathcal{P}[\{\boldsymbol{u}(\boldsymbol{R})\}] = \prod_{\boldsymbol{R}} p[\boldsymbol{u}(\boldsymbol{R})] \ . \tag{1}$$

The statistical properties of the particle distribution are thus completely determined by $p(\boldsymbol{u})$, the probability density function (PDF) for the displacement of a single particle.

In the numerical simulations reported in [1] we took a simple "box" form for the PDF. Here instead, for reasons which we will explain below, we will take $p(\boldsymbol{u})$ to be simply a mono-variate Gaussian:

$$p(\boldsymbol{u}) \equiv p(u) = \left(\frac{1}{2\pi\sigma^2}\right)^{3/2} \exp\left(-\frac{u^2}{2\sigma^2}\right) \ , \tag{2}$$

where u is the modulus of $\boldsymbol{u}$. As discussed in [1] we expect the choice of the precise form of the PDF (for any simple functional form) to have little effect on the dynamics beyond some early time transients.

We use the term *shuffling parameter*, denoted Δ, for the rms deviation of $\boldsymbol{u}$, i.e.,

$$\Delta^2 = \int_{\mathbb{R}^3} u^2 \ p(\boldsymbol{u}) \, \mathrm{d}^3u = 3\sigma^2 \ . \tag{3}$$

We will usually use this quantity expressed in terms of the lattice spacing, which we denote by ℓ. We thus define the *normalized shuffling parameter* $\delta \equiv \Delta/\ell$. This is convenient because it is actually the value of δ alone which characterizes the gravitational dynamics of an infinite SL [1]: any two SL with the same δ but different ℓ (and Δ) are equivalent up to a scale transformation which is irrelevant for (scale-free) gravity. We note that in the limit $\delta \to 0$ the particle distribution remains a perfect cubic lattice, while for $\delta \to \infty$ it becomes a Poisson distribution.

B. Characterization of particle distributions

We now discuss briefly the different statistical quantities which we will use to characterize the SL initial conditions and the evolved particle distributions. For further details, we refer the reader to [1, 13, 14].

For any distribution of N particles in a volume V, we can define the microscopic mass density as

$$\rho(\boldsymbol{x}) = \sum_{i=1}^{N} m_i \delta_{\mathrm{D}}(\boldsymbol{x} - \boldsymbol{x}_i) \tag{4}$$

where $\boldsymbol{x}_i \in V$ is the position of the ith particle, of mass m_i, and δ_{D} is the Dirac delta function. For the case of infinite systems with a well defined non-zero mean density ρ_0, which is that we consider here, it is convenient to define the density contrast $\delta_\rho(\boldsymbol{x}) = [\rho(\boldsymbol{x}) - \rho_0]/\rho_0$.

The two-point correlation properties can then be characterized by the reduced two-point (density-density) correlation function

$$\tilde{\xi}_\rho(\boldsymbol{r}) = \langle \delta_\rho(\boldsymbol{r} + \boldsymbol{x}) \delta_\rho(\boldsymbol{x}) \rangle \tag{5}$$

where $\langle \ldots \rangle$ is an ensemble average. In our simulations we treat N particles in a box of side $L = N^{1/3}\ell$ with periodic

boundary conditions. We can therefore write the density contrast as a Fourier series:

$$\delta_\rho(\boldsymbol{x}) = \frac{1}{L^3}\sum_{\boldsymbol{k}} \exp(i\boldsymbol{k}\cdot\boldsymbol{x})\tilde{\delta}_\rho(\boldsymbol{k}) \tag{6}$$

with $\boldsymbol{k} \in \{(2\pi/L)\boldsymbol{n} | \boldsymbol{n} \in \mathbb{Z}^3\}$, and

$$\begin{aligned}\tilde{\delta}_\rho(\boldsymbol{k}) &= \int_{L^3} \exp(-i\boldsymbol{k}\cdot\boldsymbol{x})\delta_\rho(\boldsymbol{x}) \\ &= \frac{1}{\rho_0}\left[\sum_i m_i \exp(-i\boldsymbol{k}\cdot\boldsymbol{x}_i) - \delta_{\rm K}(L\boldsymbol{k}/2\pi, \boldsymbol{0})\sum_i m_i\right] ,\end{aligned} \tag{7}$$

where $\delta_{\rm K}$ is the Kronecker delta function. The power spectrum[2] (PS) may then be defined (see, e.g. [14]) as

$$P(\boldsymbol{k}) = \frac{1}{L^3}\left\langle |\tilde{\delta}_\rho(\boldsymbol{k})|^2 \right\rangle . \tag{8}$$

For statistically homogeneous (i.e. statistically translational invariant) particle distributions[3], the PS and $\tilde{\xi}(\boldsymbol{r})$ are a Fourier conjugate pair and therefore contains the same information.

We estimate the PS as a function of the modulus $k = |\boldsymbol{k}|$, using simply the expression given in Eq. (7), averaged over all $\boldsymbol{k}$ in a shell in k space. We therefore denote it from now on as $P(k)$. For the real space correlations we will consider estimating $\tilde{\xi}_\rho(\boldsymbol{r})$ as defined above only in the cases where the points of the distribution considered have equal mass. Just as for the PS we will evaluate it by performing an average over all $\boldsymbol{r}$ in a radial shell. In this case we will denote it as $\xi(r)$, and it can be written[14], for $\boldsymbol{r} \neq 0$, as

$$\xi(r) = \frac{\langle n(r)\rangle_p}{n_0} - 1 \tag{9}$$

where $\langle n(r)\rangle_p$ is the radial conditional mean number density (i.e. the mean number density of particles at a distance r from an occupied point), and n_0 is the mean (unconditioned) number density. This makes its direct estimation in real space very simple, by simple pair-counting algorithms (see [1] for the explicit description of the algorithm we choose).

Besides these two quantities we will consider also, as in [1], one other one: the nearest neighbor (NN) distribution $\omega(r)$, which is simply the probability density function for a particle to have its NN at a distance r. As in [1] it is estimated in the evident way by direct counting of NN separations. The usefulness of this quantity here is that it allows one to infer [15] very useful information about the nature of the correlations at small scales, and thus indirectly about the dynamics responsible for them: if we neglect all but the two-point correlations one has the relation [14]

$$\omega(r) = \left(1 - \int_0^r \omega(s)\,{\rm d}s\right)\cdot 4\pi r^2 n_0[1+\xi(r)] . \tag{10}$$

We will use this relation (as in [1, 15]) to probe the degree to which interactions with NN particles are responsible at early times in simulations for the development of the observed correlations.

C. Gravitational clustering in a Shuffled Lattice

It is straightforward [13, 14, 16] to calculate exactly the two-point correlation properties of the SL. The PS is given by

$$P(\boldsymbol{k}) = \frac{1-|\tilde{p}(\boldsymbol{k})|^2}{n_0} + L^3\sum_{\boldsymbol{n}} |\tilde{p}(\boldsymbol{k})|^2\,\delta_{\rm K}(\boldsymbol{k}, \boldsymbol{n}\frac{2\pi}{\ell}) \tag{11}$$

where $\tilde{p}(\boldsymbol{k})$ is the Fourier transform of the PDF for the displacements $p(\boldsymbol{u})$ (i.e. its characteristic function). For the case of a Gaussian PDF as in Eq. (2), with Eq. (3), it follows that, for $\boldsymbol{k} \neq \boldsymbol{n}\frac{2\pi}{\ell}$,

$$P(\boldsymbol{k}) \equiv P(k) = \frac{1}{n_0}\left[1 - \exp\left(-\frac{\Delta^2 k^2}{3}\right)\right] . \tag{12}$$

At leading order in small k we therefore have

$$P(\boldsymbol{k}) \equiv P(k) = \frac{\Delta^2 k^2}{3n_0} . \tag{13}$$

Note that, at large $\boldsymbol{k}$, one has $P(\boldsymbol{k}) = 1/n_0$, which is simply the particle shot noise necessarily present in any distribution of point-particles.

In [1] we have studied the dynamical evolution of self-gravitating particles starting from SL initial conditions, and zero initial particle velocities. The PDF $p(\boldsymbol{u})$ used for the initial displacements off the lattice is a simple "box" form, i.e., with uniform probability inside a cube centered on the origin and oriented parallel to the lattice. The leading term in the PS Eq. (13) is in fact independent of the form[4] of $p(\boldsymbol{u})$, and thus in this approximation the initial conditions are the same.

The principal results of [1] are the following:

[2] As in [1] we use the terminology and definitions current in cosmology. In solid state physics the PS is usually called the structure factor, and normalized typically so that it is dimensionless rather than having units of volume.

[3] In a lattice, the ensemble average is defined over the set of lattices translated by an arbitrary vector in the lattice cell. In the case of the SL we have both this average and, in addition, that over all realizations of $\mathcal{P}[\{\boldsymbol{u}(\boldsymbol{R})\}]$.

[4] The only assumption about the PDF is that is has finite variance. The qualitatively different case of infinite variance PDF is treated also in [14, 16].

- Qualitatively the evolution of clustering is very similar to that observed in cosmological simulations (see, e.g., [17]), which are performed in an expanding spatial background and start from lattices with *correlated* displacements representing PS typically with $P(k) \propto k^n$ and $-3 < n < -1$. Clustered structures develop initially at very small scales ($< \ell$) and then progressively at larger and larger scales.

- The evolution is observed to be independent of the box size L until the time when the size of the non-linear structures approaches the box size itself. Thus the results of numerical simulations are interpreted as representative of the infinite volume limit (taken at constant particle density).

- The PS at small k is amplified as predicted by the linearized fluid limit for the system (see, e.g. [18]). The k below which this behavior is observed decreases with time, reflecting the propagation of the non-linear clustering in real space to larger scales.

- The temporal evolution of two-point correlation function which coincides in this case (of equal mass points) with the density-density correlation function, is well described from early times by a *spatio-temporal scaling relation*

$$\xi(r,t) = f\left(\frac{r}{\lambda(t)}\right) , \tag{14}$$

  i.e., the temporal evolution of $\xi(r,t)$ is well approximated by a simple rescaling of the spatial coordinates[5].

- After a time Δt this scaling behavior converges toward a more specific form which may be written

$$\xi(r,t+\Delta t) = \xi\left(\frac{r}{R_s(\Delta t)},t\right) \; ; \quad R_s(\Delta t) = e^{z\frac{\Delta t}{\tau_{\rm dyn}}} , \tag{15}$$

  where $\tau_{\rm dyn}$ is the *dynamical time* of the system, defined by $\tau_{\rm dyn} = 1/\sqrt{4\pi G\rho_0}$. The constant z, which one can consider a sort of *dynamical exponent* characterizing the dynamical scaling, is given by

$$z = \frac{2}{3+n} \tag{16}$$

  where n is the exponent of the power-law PS of the initial particle distribution (and therefore $z = 2/5$ for the SL). This behavior is completely analogous to what is referred to as "self-similarity" in the context of cosmology (see, e.g., [17]). It is the behavior which follows if one assumes that (i) the characteristic length scales in the system play no role in the dynamics, and (ii) the linearized fluid limit correctly describes *the evolution of* fluctuations at low amplitude. It is thus expected to be observed specifically starting from IC which are power law PS, if the clustering is independent of the lattice spacing (which is, in this case, the sole characteristic length scale). The results in [1] extend the range in which such behavior has been reported in numerical simulations in cosmology to "bluer" PS (and also to the case of a static universe).

- We have mentioned that the sole relevant parameter for the gravitational dynamics of a SL is the normalized shuffling parameter δ. In [1] different SL with a range of δ, but identical large scale amplitude of the PS, are compared. The evolution of these IC are all observed to converge to the self-similar behavior after a transient which depends on δ. These results are closely related to those described in Sec. IV below.

- In the early phase of evolution, in which non-linear clustering develops, but prior to the onset of the self-similarity, the observed two-point correlations are very well accounted for solely by two body correlations developing under the influence of NN interactions. The form of the two-point correlation function at this time is also a rather good approximation to that in the self-similar regime. Indeed, as noted above, the spatio-temporal scaling of the correlation function is a good approximation well before the self-similar regime is attained, i.e., in the transient phase we have already a temporal evolution given approximated well by Eq. (14), but with a different temporal dependence of the function $\lambda(t)$ to that in the asymptotic regime of dynamical scaling Eq. (15).

The aim of this paper, as has been discussed in the introduction, is to gain more insight into the dynamics at work in these simulations by analyzing the effect of coarse-graining the system at different times in the course of its evolution.

D. Early time evolution of a perturbed lattice

In [11, 12] we have developed in detail a general formalism for treating the evolution of self-gravitating particles perturbed off an infinite perfect lattice, in the limit that these perturbations are such that the relative displacements of the points are small compared to their separations. The treatment is thus valid up to times at which this approximation breaks down, and is applicable to any set of perturbations, whether uncorrelated (as considered here) or correlated (as in cosmological simulations). It applies equally well to the case of a static euclidean universe or a cosmological expanding one. We will make use of these results below and thus summarize them briefly

[5] In the rest of the paper we will often write $\xi(r)$ leaving the time dependence implicit.

here. We refer to this approximation to the full dynamics as *particle linear theory* (PLT).

The formalism is essentially the same as that used canonically to analyze classical phonons in a crystal in solid state physics (see e.g. [19]). One can write the equation of motion for the displacement $\boldsymbol{u}(\boldsymbol{R})$ of a particle, from its lattice site $\boldsymbol{R}$, as:

$$\ddot{\mathbf{u}}(\mathbf{R},t) = -\sum_{\mathbf{R}'} \mathcal{D}(\mathbf{R}-\mathbf{R}')\mathbf{u}(\mathbf{R}',t)\,. \tag{17}$$

where $\mathcal{D}$ is the *dynamical matrix*, which is derived by linearizing the force on a particle in the displacements relative to all other particles. For gravity we have

$$\mathcal{D}_{\mu\nu}(\mathbf{R}\neq\mathbf{0}) \;=\; Gm\left[\frac{\delta_K(\mu,\nu)}{R^3} - 3\frac{R_\mu R_\nu}{R^5}\right] \tag{18}$$

$$\mathcal{D}_{\mu\nu}(\mathbf{0}) \;=\; -\sum_{\mathbf{R}\neq\mathbf{0}} \mathcal{D}_{\mu\nu}(\mathbf{R})\,. \tag{19}$$

Defining the discrete Fourier transform on the lattice by

$$\tilde{\mathbf{u}}(\mathbf{k},t) = \sum_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}}\mathbf{u}(\mathbf{R},t) \tag{20}$$

$$\mathbf{u}(\mathbf{R},t) = \frac{1}{N}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}}\tilde{\mathbf{u}}(\mathbf{k},t) \tag{21}$$

where the sum in $\boldsymbol{k}$ is over the first Brillouin zone, i.e., for the simple cubic lattice $\boldsymbol{k}\in\boldsymbol{n}(2\pi/L)$, where $\boldsymbol{n}$ is a vector of integers each $\in]-\frac{N^{1/3}}{2}\,,\frac{N^{1/3}}{2}]$, the equation of motion Eq. (17) becomes, in reciprocal space,

$$\ddot{\tilde{\mathbf{u}}}(\mathbf{k},t) = -\tilde{\mathcal{D}}(\mathbf{k})\tilde{\mathbf{u}}(\mathbf{k},t) \tag{22}$$

where $\tilde{\mathcal{D}}(\mathbf{k})$, the FT of $\mathcal{D}(\mathbf{R})$, is a real symmetric 3×3 matrix for each $\mathbf{k}$.

The dynamical problem thus reduces to the diagonalisation of $\tilde{\mathcal{D}}(\mathbf{k})$ for each $\mathbf{k}$ in the first Brillouin zone. Labeling the three orthonormal eigenvectors $\hat{\mathbf{e}}_n(\mathbf{k})$ and their eigenvalues $\omega_n^2(\mathbf{k})$ $(n=1,2,3)$, the evolution of the displacement field from $t=0$ is given as[6]

$$\begin{aligned}\mathbf{u}(\mathbf{R},t) \;=\; &\tfrac{1}{N}\textstyle\sum_{\mathbf{k}}\sum_n \hat{\mathbf{e}}_n(\mathbf{k})\cdot\{\tilde{\mathbf{u}}(\mathbf{k},0)\cdot\hat{\mathbf{e}}_n(\mathbf{k})\cosh[\omega_n(\mathbf{k})t]\\ &+\tfrac{1}{\omega_n(\mathbf{k})}\dot{\tilde{\mathbf{u}}}(\mathbf{k},0)\cdot\hat{\mathbf{e}}_n(\mathbf{k})\sinh[\omega_n(\mathbf{k})t]\}e^{i\mathbf{k}\cdot\mathbf{R}}\end{aligned} \tag{23}$$

Thus given the initial displacements and velocities, the dynamical evolution is solved.

The solution of the diagonalisation problem is numerically straightforward. Details of it, and the results for the eigenvalues and eigenvectors, are given in [11]. In this paper the domain of validity of this perturbative treatment has also been investigated using numerical simulations. Starting from (i) an SL initial condition and (ii) from a perfect lattice configuration with correlated perturbations corresponding to a PS $\sim k^{-2}$, it is found that the treatment traces very well, in both cases, the full evolution of the system (i.e. with the full gravitational interaction) until the average relative displacement approaches the lattice spacing.

E. The Zeldovich Approximation

Keeping the mean mass density ρ_0 fixed, and taking the limit $\ell\to 0$, the eigenvectors and eigenvalues at any given fixed k simplify: one obtains a single non-zero eigenvalue *independent of* k, $\omega^2(\mathbf{k}) = 4\pi G\rho_0 \equiv \tau_{\rm dyn}^{-2}$, associated to a longitudinal mode $\hat{\mathbf{e}}(\mathbf{k})=\hat{\mathbf{k}}$, and two transverse modes (i.e. $\hat{e}(\mathbf{k})\cdot\hat{\mathbf{k}}=0$) with zero eigenvalues. If we consider the case that the perturbations to the lattice are of long wavelength, i.e., that there are perturbations only with $k\ell\ll 1$, we obtain from Eq. (23) that there is an asymptotic attractor solution for large times ($t\gg\tau_{\rm dyn}$), which may be written

$$\mathbf{u}(\mathbf{R},t) = \exp(t/\tau_{\rm dyn})\,\mathbf{q}(\mathbf{R})\,, \tag{24}$$

where $\mathbf{q}(\mathbf{R})$ is the time independent curl-free (irrotational) vector field:

$$\mathbf{q}(\mathbf{R}) = \frac{1}{N}\sum_{\mathbf{k}}\left\{(\tilde{\mathbf{u}}(\mathbf{k},0)\cdot\hat{\mathbf{k}}) + \tau_{\rm dyn}(\dot{\tilde{\mathbf{u}}}(\mathbf{k},0)\cdot\hat{\mathbf{k}})\right\}\cdot\hat{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{R}} \tag{25}$$

Using Eq. (22) it is easy to show that the gravitational field can be written, in the same long-wavelength limit, as

$$\mathbf{g}(\mathbf{R},t) = \tau_{\rm dyn}^{-2}\mathbf{u}(\mathbf{R},t) \tag{26}$$

so that it follows that the asymptotic solution can be written as

$$\mathbf{u}(\mathbf{R},t) = \exp(t/\tau_{\rm dyn})\tau_{\rm dyn}^2\mathbf{g}(\mathbf{R},0)\,, \tag{27}$$

$$\dot{\mathbf{u}}(\mathbf{R},t) = \exp(t/\tau_{\rm dyn})\tau_{\rm dyn}\mathbf{g}(\mathbf{R},0)\,, \tag{28}$$

i.e., both the displacement and velocity of each particle is expressed solely in terms of the gravitational field acting on it at the initial time [7].

This approximation corresponds to one introduced by Zeldovich [20], for the evolution of fluid elements in a pressure-less self-gravitating fluid away from a perfectly

[6] We give here the formulae for the specific case of a static euclidean universe. The general result for cosmological backgrounds is given also in [11].

[7] The approximation is thus sometimes described as "ballistic". This, however, is a misnomer as the approximation incorporates in fact the collective effect of the motion of all particles on one another: the approximation is non-trivial precisely insofar as it extends well beyond the regime of a ballistic approximation.

uniform state, in an expanding universe. It can be formally derived [21] as an approximation through a perturbative treatment of the fluid equations in the Lagrangian formalism[8].

Note that using the continuity equation for small displacements $\mathbf{u}$ applied to elementary volumes in a continuous medium one obtains that the density fluctuations $\delta\rho$ with respect to the mean density ρ_0 are given by $\delta\rho \approx -\rho_0 \nabla \cdot \mathbf{u}$. Linear amplification of small density fluctuations is thus associated with the growth of displacements described by Eq. (24), and indeed it was on the basis of this observation that Zeldovich proposed his ansatz. The power of the Zeldovich Approximation (ZA) is that its domain of validity extends well beyond that of linear (Eulerian) perturbation theory of the self-gravitating fluid. In fact it extends up to "shell-crossing", when fluid elements contract in one direction to produce density singularities. In terms of our particle treatment this shell-crossing corresponds to the approach of particles to one another.

In cosmological simulations the ZA — in the form of Eq. (27) adapted to an expanding universe — is used to set up initial conditions: it gives a prescription for both the displacements and velocities of the particles off the lattice (considered as the centers of fluid elements) once the input PS is given, as any realization of the latter gives the gravitational field through the Poisson equation. Our initial conditions are not equivalent to this as the displacement field applied to the lattice at the initial time is not, in general, curl-free as in the ZA (since the gravitational field is the divergence of a scalar field). We can, however, use the ZA to determine the velocities since Eqs. (27) and (28) imply

$$\dot{\mathbf{u}}(\mathbf{R}, 0) = \frac{\mathbf{u}(\mathbf{R}, 0)}{\tau_{\rm dyn}} \,. \tag{29}$$

This is the prescription we will adopt here (rather than the zero initial velocities of [1]). It is a very natural one as it uses the single characteristic time scale of the fluid evolution to define the velocities. Further the choice is essentially equivalent to that in cosmological simulations, as it means that the PS at small k (i.e. in the fluid limit) evolves exactly as in this context, growing in proportion to the square of the growing mode of linear theory $e^{t/\tau_{\rm dyn}}$ [9], and likewise the displacement and velocity fields in the fluid limit:

$$\mathbf{u}(\mathbf{R}, t) = \exp(t/\tau_{\rm dyn}) \mathbf{u}(\mathbf{R}, 0) \tag{30}$$

$$\dot{\mathbf{u}}(\mathbf{R}, t) = \exp(t/\tau_{\rm dyn}) \dot{\mathbf{u}}(\mathbf{R}, 0) \,. \tag{31}$$

[8] The more general form of the solution at leading order in this perturbative scheme can in fact be derived using the formalism described here. See [11, 12] for details.

[9] This is true because in the small k (fluid) limit only the curl-free component of the displacement field contributes to the density fluid fluctuations. This follows from the relation $\delta\rho \approx -\nabla \cdot \mathbf{u}$, which is valid in this limit. See [11, 12] for the exact expressions and analysis of the fluid limit.

III. PARTICLE COARSE-GRAININGS

1. *Defining a coarse-graining (CG)*

In this section we describe precisely how we construct the coarse-grained particle distributions we study.

To coarse-grain a distribution of massive point particles:

- 1. We define a set of finite non-overlapping equal volume cells covering all space, i.e., a tiling of three dimensional space with equal volume tiles. We consider here, for simplicity, only the case that these cells are those of a *simple cubic lattice*, with lattice spacing which we denote ℓ_{CG}. Since the system studied here is evolved assuming periodicity in a cubic box of side L, we choose $\ell_{CG} = L/p$ where p is a positive integer.

- 2. We ascribe one particle to each cell. The mass of this (point) particle is equal to the sum of the masses of the particles which are inside the given cell in the distribution being coarse-grained, and its position and velocity coincide with those of the center of mass and of these particles.

We study here an initial particle distribution which is a SL or a SL evolved under gravity, in which all the points are ascribed equal mass. In general the point particles in the coarse-grained distribution just defined will not have the same mass since the number of points in the coarse-graining cells will not necessarily be equal. However in the cases which we consider in this paper, these particle mass fluctuations introduced by the CG will be very small: (i) we will consider CG particle distributions at scales ℓ_{CG} which are large compared to the typical displacements of particles from their initial lattice position, and (ii) we will consider CG cells with

$$\ell_{CG} = \alpha \ell \,, \alpha \in Z^+ \tag{32}$$

which are perfectly aligned with the original lattice, and therefore each contain exactly the same number of points of the unperturbed lattice. To be more precise in quantifying the importance of these mass fluctuations, it will be useful to make the following distinction: one can consider the above described CG as a "Eulerian" CG, in contrast to a "Lagrangian" CG. In the latter case we modify step 2 in the above definition to read:

- 2'. We ascribe one particle to each cell. The mass, position and velocity, of this particle are those of the centre of mass of the points which were, before the application of the initial random displacements, on lattice sites inside the given cell.

In this case — with CG cells aligned with the original lattice with ℓ_{CG} as in Eq. (32) — the CG particle distribution retains evidently the equal mass property of the original distribution.

2. Some properties of coarse-grainings

A few basic results which follow trivially from these definitions are important to note:

- A Lagrangian CG of a SL is itself an SL.
- A Lagrangian CG of a SL with Gaussian PDF is itself a SL with Gaussian PDF.
- An Eulerian CG of a SL converges, in the limit of small displacements, to the Lagrangian CG on the same grid.

The first result is very simple to see. The position of the particles in the Lagrangian CG can be written as

$$\mathbf{x}_I = \mathbf{R}_I + \mathbf{u}(\mathbf{R}_I)\,, \tag{33}$$

where

$$\mathbf{R}_I = \frac{1}{\alpha^3}\sum_i^{(I)} \mathbf{R}_i \quad , \mathbf{u}(\mathbf{R}_I) = \frac{1}{\alpha^3}\sum_i^{(I)} \mathbf{u}(\mathbf{R}_i) \tag{34}$$

and the sums are over the α^3 points i whose original lattice sites are inside the I-th cell of the CG lattice. $\mathbf{R}_I$ is simply the center of the new CG lattice cell, and, since the $\mathbf{u}(\mathbf{R}_i)$ are uncorrelated with variance Δ^2, we have

$$\langle \mathbf{u}(\mathbf{R}_I)\cdot\mathbf{u}(\mathbf{R}_J)\rangle = \delta_K(I,J)\frac{\Delta^2}{\alpha^3} \equiv \delta_K(I,J)\Delta^2_{CG}. \tag{35}$$

Thus the CG particle distribution is a SL, with variance reduced by α^3. The PDF of the displacements in the SL can be calculated given that of the original particle distribution, but in general has a different functional form. As noted above, however, for the case of a Gaussian the PDF is the unchanged, because of the fundamental property of stability of a Gaussian (see, e.g., [14]). For the particle distributions we considered here as IC — SL with small amplitude Gaussian displacements — the Eulerian CG is practically equivalent to the Lagrangian one.

We note one further important property of the CG particle distributions defined in this way: *the PS of mass density fluctuations in the original particle distribution and that in the CG constructed from it converge at scales larger than* $\sim 2\pi/\ell_{CG}$. This follows from a well-known argument due to Zeldovich [18, 22, 23]. This argument[10] states that the perturbations to a mass distribution introduced by moving matter around on a finite scale, while preserving locally the center of mass, lead to a modification to the PS at small k (i.e. smaller than the inverse of the characteristic length scale) which is proportional to k^4. Since, as we have seen above, a SL has a PS which is proportional to k^2 at small k, the result follows.

For the case of a CG on a SL, we can in fact derive this result directly from the expressions Eqs. (12) and (13) given above: for $\ell_{CG} = \alpha\ell$ we have that $n_0 \to n_0/\alpha^3$ and, we have seen, $\Delta^2 \to \Delta^2/\alpha^3$. Thus the leading k^2 term of the PS, given in Eq. (13), is invariant under the coarse-graining, while the next term, proportional to k^4, is not.

This convergence of the PS of the particle distribution and its CG quantifies the sense in which the CG leaves invariant the large scale properties of the particle distribution. Since a continuous fluid-like description of such systems and their dynamics is expected to be valid for precisely such scales, one would anticipate that this description should coincide for both systems. Indeed this is one of the underlying motivations for our study of this kind of CG.

[10] We use the word "argument" because the heuristic derivation given by Zeldovich — and other authors in the cosmological literature — fall well short of a proof. See [24] for a discussion of this issue and some explicit constructions which illustrate the result.

Name	$N^{1/3}$	L	ℓ	Δ	δ	m/m_{64}
SL64	64	1	0.015625	0.00195	0.125	1
$CG32_0$	32	1	0.03125	0.00069	0.0221	8
$CG16_0$	16	1	0.0625	0.00024	0.0039	64

TABLE I: Details of the three sets of initial conditions studied in this section. SL64 is the original (or "mother") particle distribution and $CG32_0$ and $CG16_0$ are the two "daughter" particle distributions obtained by CG it as described in the text. N is the number of particles and L is the box size (arbitrarily set to unity). The mass m of the particles is given in units of that in SL64, i.e., m_{64}. The mass density is of course the same by construction in all the particle distributions, $\rho_0 = Nm/L^3 = 1$.

IV. COARSE-GRAINING OF THE INITIAL CONDITIONS

In this section we study the evolution from a SL initial condition, and from CGs defined on this initial particle distribution. We first present and analyze our numerical results, and then analyze what can be learned from them about the non-linear clustering in these systems. Specifically we will see that the non-linear two-point correlation function, which develops in a given such system, can be well reproduced, over a significant range of time and spatial scale, in a CG of the system evolving in a regime in which the relevant dynamics are driven by two body interactions of NN pairs. There is, however, a small discrepancy in the amplitude of the correlation function which we show can be quantitatively understood using PLT (and which is an effect of discreteness, i.e., due to the different particle density in the simulations). We then show that these, and other, differences between the *mother* (i.e. the original particle system) and *daughter* (i.e. the CG particle distribution) systems will increase,

without limit, as the parameter $\alpha = l_{CG}/\ell$ does. This then leads us naturally to the CG presented in the following section.

As our original ("mother") system we take here, and in the next section, the following IC: a simple cubic lattice with $N = 64^3$ particles, to which we apply a shuffling with Gaussian PDF and normalized shuffling parameter $\delta = 0.125$, i.e., as given by Eqs. (2) and (3) with $\Delta = 0.125\ell$ (and $\ell = L/N^{1/3}$). We will call this initial condition SL64.

We consider then also two other IC which are Lagrangian CG as defined above: $\mathrm{CG32}_0$ and $\mathrm{CG16}_0$, defined respectively on the 32^2 and 16^3 sub-lattice of the original lattice (i.e. with $\alpha = 2$ and $\alpha = 4$) respectively. Since δ is small this CG is, as we have noted above, in practice equivalent to the Eulerian CG[11]. Given that $\ell_{CG} = \alpha\ell$ and $\Delta^2_{CG} = \Delta^2/\alpha^3$, we have

$$\delta_{CG} \equiv \frac{\Delta_{CG}}{\ell_{CG}} = \frac{\delta}{\alpha^{5/2}} \tag{36}$$

which explains the values of δ which appear in Table I. Further, in order to keep the mass density ρ_0 fixed, we take

$$m_{CG} = \frac{m}{\alpha^3}\,. \tag{37}$$

where m, the mass of particles in the original SL, is (arbitrarily) normalized to unity in SL64.

The initial velocities of SL64 are fixed using Eq. (29). In the two CGs they are fixed, as described above, by the condition that the CG points are ascribed the velocity of the center of mass of the set of points they represent. It is evident that this is in fact equivalent (for the Lagrangian CG) to also using Eq. (29) on the CG points directly.

A. Results of numerical simulations

The numerical code used for our simulations is, as in [1] and [11], the publically available GADGET code [25]. It uses a tree algorithm for the calculation of the gravitational force. The singularity at $r = 0$ in the latter is regularized, for numerical efficiency, at very small scales, i.e., well below the initial inter-particle spacing ℓ. The simulations here (as in [1]) use a "smoothing parameter" $\varepsilon = 1/15 \cdot 1/64 \approx 0.0010$ in the units used in Table I, i.e., about one fifteenth of the smallest *initial average inter-particle distance* in our simulations [12]. We assume here, as in [1] and indeed generically in all such cosmological simulations, that the results we consider as physical, starting from separations a few times larger than ε, are essentially independent of this choice of ε. As in [1] we have tested this assumption, and found it to be good, by resimulating identical initial conditions changing only ε to considerably smaller values.

In Fig. 1 are shown snapshots of two-dimensional projections of the simulations starting from SL64, $\mathrm{CG32}_0$ and $\mathrm{CG16}_0$. Here, and in the rest of this paper, *time is given in units of dynamical time* $\tau_{\rm dyn}$. We stop our analysis at $t = 8$ as for longer times finite size effects begin to be important as the size of the structures formed approach the size of the box. As time evolves structures first form in each simulation at small scales — well below the initial grid size ℓ — and then subsequently at larger scales. Clustering develops first in SL64, and only later in its CG. While no similarity in the fluctuations is *visually* discernible when the particle distributions are grid-like, in the evolved configurations one sees clearly that the structures which formed in the CGs trace approximately those in the "mother" distribution.

In Fig. 2 we show the evolution of the two-point correlation function $\xi(r)$ at the times indicated. These plots reflect in a quantitative manner the information gleaned from the visual inspection of the snapshots in the previous figure. The three systems show initially no discernible similarity: the real space correlation function is in each case dominated by the features of the different underlying lattice. Strong clustering (with $\xi \gg 1$) develops first at small scales in the SL64 simulation, and then subsequently at larger scales. When the coarse-grained systems start to evolve a clustered regime, their correlation functions start rapidly to resemble strongly those in the "mother" system. We note, however, that there is a transient time in which the coarser system has a correlation function resembling that of the "mother system", but of a slightly lower amplitude. This "lag" of the coarser system is here manifest in the plot at $t = 6$. We will return to this point below.

In Fig. 3 is shown the evolution of the PS in the three simulations at different times steps, with k in units of the Nyquist frequency $k_N = \pi/\ell$ of SL64. The three curves at $t = 0$ correspond to the theoretical PS given in Eq. (11), with the appropriate Gaussian form $\tilde{p}(k)$ and value of the shuffling variance. The agreement of the PS at small k discussed above is clearly seen (and one can verify the accuracy of the k^2 behavior). The structure of peaks, which is different in each initial condition, comes from the second term in Eq. (11)[13]. Note that while the

[11] The probability p_{out} that any given particle of the initial $N = 64^3$ lattice is displaced by its shuffling so that it falls outside the CG cell of the 32^3 CG can be written as $p_{out} = 1 - [\sqrt{3/2\pi}\int_{-1/2\delta}^{3/2\delta} e^{-3x^2/2}dx]^3$. For $\delta = 0.125$ this corresponds to $p_{out} \sim 10^{-12}$. Thus $Np_{out} \ll 1$.

[12] For this code [25] this means that the force is the exact gravitational force for separations greater than 2ε. For the precise functional form of the regularization below this scale, see [25].

[13] Note that not all the Bragg peaks in the PS [cf. second term in Eq. (11)] are visible in the plot.) Indeed the first Bragg peak in SL64 is at $k = 2k_N$, while the first one in $\mathrm{CG32}_0$ and $\mathrm{CG16}_0$ are at $k = 4k_N$ and $k = 8k_N$ respectively. The reason is that the PS is calculated using a sampling of $\boldsymbol{k}$, and the peaks are so narrow that they are missed.

FIG. 1: Snapshots at times 0 (top line), 3, 6 and 8 (bottom line) of SL64 (left column), $CG32_0$ and $CG16_0$ (right column). What is shown is a projection on the x–y plane of a slice of thickness 0.6 along the z axis.

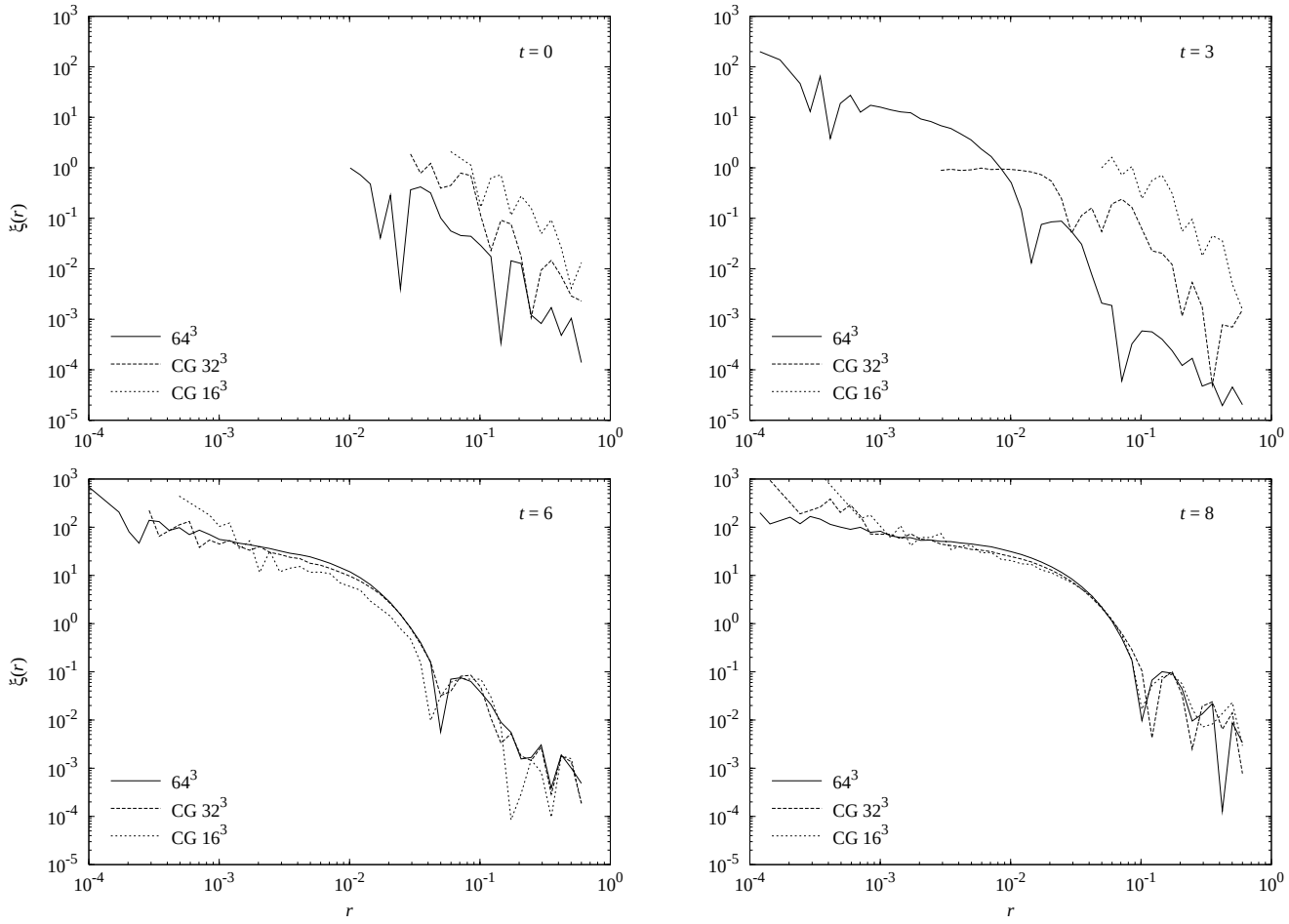

FIG. 2: Evolution of $\xi(r)$ in SL64 and in $CG32_0$ and $CG16_0$. The times are given in the plots.

differences between the distributions at the initial time manifest themselves in the PS only at large k, we have seen that the initial correlation functions are very different at all scales[14]. At $t = 3$ this peak structure is still visible only in the $CG16_0$ simulation, corresponding to the fact that the lattice structure is still present in this simulation which has evolved little non-linear clustering at this time. The relative difference between two other simulations is now less than at the initial time, showing again that once structures develop in the "daughter" particle distribution they trace those in the original distribution quite accurately. At later times this behavior is also seen for the $CG16_0$ simulation, as it "catches up" with the other ones.

These observations are, to a first approximation, very much in line with the qualitative picture of the evolution of clustering widely accepted in the theory of structure formation perturbations in cosmology (see, e.g., [18], or [27] for a concise summary): non-linear gravitational evolution transfers power from larger to smaller scales (by collapse) so efficiently that, for initial PS with $n < 4$ (as is the case here, $n = 2$), the clustering amplitudes in the non-linear regime at a given time depend essentially only on the initial power in modes corresponding to larger scales. Thus what is relevant to recover the non-linear evolution of the system is to include this initial power at larger scales[15]. The coarse-grainings we have

[14] This "localisation" of the discrete characteristics of the distributions in reciprocal space, and "delocalisation" in real space, is also a feature of the initial conditions of cosmological simulations. See [26] for a detailed discussion.

[15] See, e.g., [28] for a numerical study of this issue in cosmological simulation, and [29] for a widely used phenomenological ansatz for "reconstruction" of the non-linear PS from the initial condi-

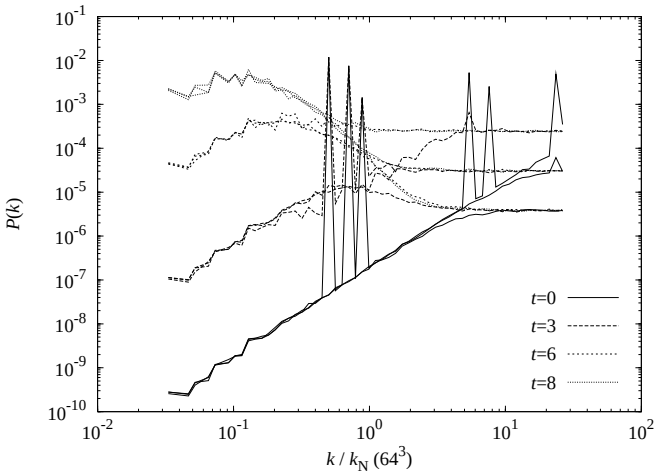

FIG. 3: Evolution of the PS in SL64, $CG32_0$ and $CG16_0$. Times are as indicated in the legend. The x-axis is normalized to k_N in SL64. At each time the different systems can be most easily distinguished by looking at the PS at large k, which approaches cowards $1/n_0$. The peaks which are evident in in the PS are the Bragg peaks of the lattice, which progressively disappear as the lattice structure is washed out by the evolution.

applied conserve, by construction, the initial power at larger scales, and thus can begin to reproduce clustering correctly at scales smaller than the CG scale once the smallest scales with the "correct" power go non-linear. More qualitatively, the approximate criterion used in cosmology to determine when non-linear evolution should set in, adapted to the present case of a static universe, is

$$k^3 P(k) e^{2t/\tau_{\rm dyn}} \sim 1\,, \tag{38}$$

i.e., the fluctuations of wavenumber k satisfying this relation at a given time are those "going non-linear" according to the linearized fluid theory at that time [16]. Assuming that the coarse-graining procedure leaves only initial fluctuations in reciprocal space below the Nyquist frequency, the largest k available to go linear, which is therefore the first one to do so (since $P(k) \propto k^n$ with $n > -3$), is the Nyquist frequency, $k_N = 2\pi/\ell_{CG}$. The latter decreases, and therefore non-linearity sets in later, as ℓ_{CG} increases. To determine the time scale this argument predicts, it is convenient to note that, using Eqs. (13) and (36), the criterion Eq. (38) with $k = k_N$, may be rewritten

$$\delta^2_{CG} e^{2t/\tau_{\rm dyn}} \sim 1\,. \tag{39}$$

Thus the time scale predicted for the first non-linear structures is just the time of fall of neighbouring particles on one another, *calculated in the fluid limit* (i.e. using the ZA as in Eq. 27.). We will return to the question of the accuracy of this prediction below, and explain that it will be a good approximation if discreteness effects up to this time (i.e. when particles first approach one another) are small. For the numerical simulations here this is indeed a reasonable first approximation, because the duration of this period is not too long (at most a few dynamical times).

To the extent that the evolution of the measured two point quantities agrees (after some time), one can conclude that this evolution can depend only on quantities which are the same in all the systems. Specifically, as we have just discussed, it can (and does) depend on the amplitude of the initial fluctuations at the relevant larger scales, but it *does not depend on the particle density*. From this latter fact one might be tempted to conclude that the dynamics of the system should be well described by one in which the density fields, for example, are smooth functions, i.e., by a description in which the discrete nature of the system is of no relevance[17]. Independence of measurable quantities of the particle density does not, however, establish such a conclusion. Indeed we will see now that, on the contrary, a description of the dynamics in a manifestly particle-like (i.e. discrete) way captures essential aspects of the evolution.

We consider now more closely both the differences and similarities between the evolved simulations. We note in particular that the CG systems trace very well the two point correlations in the original system, already at a time when a very simple model for the development of these correlations, in terms of two body interactions, is valid. There is, however, a small discrepancy in the amplitudes of the correlation functions, which we explain. This leads us naturally to consider the extrapolation of these results to the case that the CG scale becomes very large compared to the inter-particle distance in the original particle distribution.

B. Non-linear correlations at early times

In the results above we have seen clear evidence for the convergence of the correlation properties of the evolved coarse-grained IC toward those of the evolved "mother" particle distribution. There is, however, a range of time scales in each case in which there are already significant non-linear correlations in the CG simulation, but the agreement, notably in the amplitude of the correlation

tions.

[16] The quantity $k^3 P(k)$ is equal, up to a numerical factor, to the variance of mass fluctuations averaged over a spherical Gaussian window function of radius $\approx 2\pi/k$ cf. [18]. We have adapted the criterion here to the case of a non-expanding universe, by using the linear theory growth factor appropriate for this case.

[17] We have in mind, specifically, the description of the dynamics by a Vlasov-Poisson system of equations. In cosmology the aim of N-body simulations is to reproduce the evolution of this system of equations. We will discuss this question in greater detail in the conclusions section.

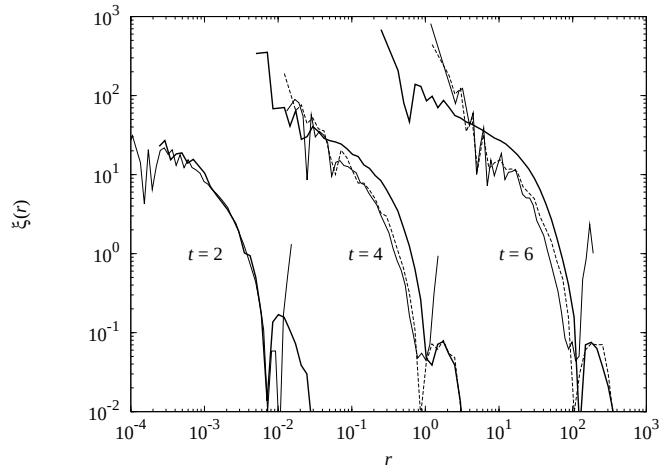

FIG. 4: Reconstruction of $\xi(r)$ using the NN distribution $\omega(r)$ in Eq. (10). At time $t = 2$, it is applied to SL64: the thick line represents the directly measured $\xi(r)$, while the thin line is the reconstruction with $\omega(r)$. The chosen times $t = 4$ and $t = 6$ for the other two plots correspond to times a little before the correlation function of $CG32_0$ and $CG16_0$ catch up with SL64. In the two cases, we have plotted $\xi(r)$ in SL64 (thick lines), $\xi(r)$ in the CG system (dashed line) and its reconstruction with $\omega(r)$ (thin continuous line). This allows one to see that at these times, when the first non-linear correlations are created in all these systems, they are well accounted for purely by correlation of NN pairs, which develop through the interaction of such pairs. For clarity, the data at the two later times have been rescaled along the x-axis (by a factor 50 and 50^2 respectively).

function, is not so good. This discrepancy can be understood by analyzing the dynamics giving rise to these correlations at early times.

We have shown in [11] that the evolution at early times — when the displacements are small compared to the lattice spacing — is very well described by the perturbative approach (PLT) described briefly in Sect. II D above. It breaks down when pairs of particles start to approach one another. In the immediately subsequent phase strong correlations (with $\xi \gg 1$) begin to develop at small scales, and it is natural to hypothesize that an important role is played at this time by the interaction of particles with their NN. A simple way to test for this is, as explained in Sect. II B above, is to see whether the relation Eq. (10) holds. To do this we "reconstruct" the two-point correlation function $\xi(r)$ from the measured NN distribution using this relation, and compare with the measured $\xi(r)$ at each time. The results are shown in Fig. 4. Inspection shows that in each of the simulations the relation holds to a very good approximation. The times at which we have chosen to plot the comparison in each case is the largest time at which the agreement is observed to be good. This time increases as the IC becomes coarser because, as discussed above, the phase in which PLT is valid increases as the CG scale does. Therefore the time of switching on of binary collapse becomes correspondingly larger.

The CG systems thus develop significant non-linear correlation in the phase in which the relation Eq. (10) holds. We infer that the correlation measured is essentially the result of two body correlations developing through the interaction of particles with their NN. What is very striking, as it has been pointed out in [1, 15], is that the two-point correlation function traces very well, at least *in its shape*, that in the mother simulation, even though in the latter the dynamics are at this time no longer NN dominated. At these times in fact the mother simulation is evolving in the self-similar manner described in [1], for which the temporal scaling [i.e. the dynamical exponent given in Eq. (15)] can be determined from a fluid-like description of the system[18]. Thus the complex many body dynamics of the mother system (whether mean field or not) is well approximated, at the relevant scales, as particle-like "blobs" each moving under the effect of the single nearest "blob".

C. Extrapolation to large ℓ_{CG}/ℓ

There is, however, as we have noted also in the discussion of Fig. 2, a small (but discernible) discrepancy between the *amplitude* of the correlation function in the CG system and that in the original one. The CG systems thus appear to lag behind for a time, before "catching up" subsequently with the mother system. Thus the dynamics of the formation of the correlation in the mother system, in the asymptotic self-similar regime, appears to be very well represented by that of CG particles, but these particles fall toward one another by NN interactions slightly slower than the particle-like "blobs" of the original particle distribution.

This difference can in fact be understood from the perturbative approach (PLT) of [11], which we outlined above in Sec. II D. For most modes of the displacement field applied to a simple cubic lattice, the largest eigenvalues characterizing growth are in fact *smaller* than the (single) exponent characterizing the fluid limit, given by the ZA as in Eq. (30) for our initial conditions. Thus the evolution of displacements is, typically, *slower* in the discrete particle distribution (described accurately by PLT) than in the fluid limit[19]. Further the discrepancy in any given quantity in the two treatments becomes more important as the time increases (so long as PLT itself remains valid). Specifically, for example, the criterion Eq. (39), derived above in the fluid (ZA) limit, for the

[18] We recall that the scaling is inferred from that of the large scale fluctuations in the model treated in the linearized fluid limit, cf. [1].

[19] We write "typically" because there are a very small fraction of modes, in a simple cubic lattice, which grow slightly *faster* than in the fluid limit. Given their small number, and the small difference in the growth rate, they are significant in the average only at *very* long times.

time of onset of the phase of development of strong nonlinear correlations (dominated, at first, as we have seen by NN interactions) can be calculated more accurately using PLT. The result can be written schematically as

$$\delta_{CG}^2 e^{(1-\eta)2t/\tau_{\rm dyn}} \sim 1\,. \tag{40}$$

where η is different from zero, and typically positive[20]. Thus the discrepancy between the fluid prediction of Eq. (39) and that of PLT Eq. (40) will increase as δ_{CG} decreases, i.e., as α increases [cf.Eq. (36)].

It follows that if we extrapolate the coarse-graining to arbitrarily large α, an arbitrarily large difference in the evolution of the CG daughter and that of the original distribution will result[21]. Such behavior could, in priciple, be observed, of course, in numerical simulations like those performed here, if the initial δ is sufficiently small. In practice this is not numerically feasible as it requires unattainable accuracy on the force[22]. We note also that the small time-lag effect which we have identified here in our simulations was not seen in the results of [15], because the SL simulations considered started from larger values of δ (larger, specifically than that in $\mathrm{CG16}_0$, cf. Table I).

V. COARSE GRAINING AT A FINITE TIME

We have seen in the previous section that the CG on a grid of the SL initial conditions, i.e., of the system at the initial time, leads to an evolution which diverges further and further from that of the original system as the ratio of the CG scale to the scale of the original grid increases. The reason is that the CG system of particles evolves for a longer time in the regime in which discreteness effects induced by the lattice accumulate.

In this section we define and study a different CG which, by construction, avoids this problem: instead of coarse-graining the initial condition we coarse-grain the original system *when it has evolved for a finite time*. The idea is simple: we choose the time at which we coarse-grain the system at a given scale by a prescription which means that, approximately (in the sense discussed in what follows), the parameter δ_{CG} (i.e. the one point variance of the displacements of the CG particles off their lattice) is *equal* to that of the original particle distribution. The CG system will therefore spend approximately the same amount of time in the regime of validity of PLT as the original system.

A. Definition of coarse-graining at a finite time

As in the previous section we consider coarse-graining a particle distribution with lattice spacing ℓ using a CG grid which is also a lattice with lattice spacing $\ell_{CG} = \alpha\ell$, where α is an integer. The prescription we use for the time t_{CG} at which we perform the coarse-graining is

$$t_{CG} = \tau_{\rm dyn}\frac{5}{2}\ln\alpha\,. \tag{41}$$

The reason for the choice Eq. (41) is simple : *if* the Zeldovich approximation is assumed to be exact at the scale of the coarse-graining, the CG system is then exactly a SL *with the same value of the normalized shuffling parameter* δ. Thus the CG at time t_{CG} is related to the original system at $t = 0$ by a simple rescaling of all length scales[23]. Let us see that this is the case.

When the ZA is valid, in the form as given in Eq. (27), it follows that all particle displacements are simply amplified in time by the factor $e^{t/\tau_{\rm dyn}}$. Thus a SL remains a SL, with a growing variance of its random uncorrelated displacements. If we perform a Lagrangian CG at time t we therefore obtain a SL with normalized shuffling δ_{CG} given as Eq. (36), i.e.,

$$\delta_{CG} = \frac{\delta(t)}{\alpha^{5/2}} = \frac{\delta(0)e^{t/\tau_{\rm dyn}}}{\alpha^{5/2}}\,. \tag{42}$$

Taking the prescription Eq. (41) for the time at which the CG is performed, we thus have $\delta_{CG} = \delta(0)$. Further we recall that in the ZA the velocities scale in proportion to the displacement [cf. Eqs. (27) and Eq. (28)]. Thus the velocities in the CG particle distribution also simply rescale those in the original distribution in proportion to α. The CG system is thus (statistically) identical to the original system up to a rescaling of spatial variables of the original system $\vec{x} \to \alpha\vec{x}$ [24].

In reality of course the ZA is not exact. Above we have assumed in fact that it is exact for the evolution *of each particle* of the mother distribution until an arbitrary time

[20] In [11] the full gravity, PLT and fluid evolution is calculated for the average relative displacement between particles initially on neighboring sites of a SL configuration, and the "lag" effect, described approximately by Eq. (40) with $\eta > 0$, is manifest (see e.g. Fig. 8 of [11]). At *very* long times the sign of the effective η will change as the few modes which grow more rapidly than in the fluid (see preceeding footnote) come to dominate.

[21] Indeed for asymptotically large α one will obtain completely different qualitative behaviours of the two systems, as the small number of eigenmodes which grow faster than in the fluid limit will dominate. The eigenmode with the very largest eigenvalue (about ten percent larger than the fluid one) corresponds to adjacent pairs of planes of the lattice falling toward one another [11]. In the limit that δ is sufficiently small that such a mode can collapse before any other one, there will be an imprinted long range order in the structures formed which is sheerly an artifact of the CG lattice.

[22] In [11], however, related oscillating modes of the lattice have been simulated using a simplified code exploiting the symmetries of certain modes.

[23] This is for the infinite lattice and exact gravity. It is thus implicit here that we consider the regime where finite box size effects, and effects depending on the finite softening ε may be neglected.

[24] We assume as the previous section that we have a Gaussian PDF.

(since t_{CG} increases arbitrarily as ℓ_{CG} does). Clearly this is not even a good approximation in general. We need, however, only to make a much weaker assumption to recover the result (that the *finite-time* CG is a rescaling of the mother SL): all we require in fact is that *the center of mass of the CG cell moves according to the ZA until* t_{CG}. Thus, specifically, we require only that the ZA be a good approximation when we neglect the gravitational forces acting between particles within the same CG cell (since such forces do not affect the center of mass motion).

We have seen that the ZA can be derived as an appropriate limit of the full particle evolution using PLT. This latter treatment is of course only valid until the time at which particles approach one another for the first time. We can, however, modify it to understand the nature of the approximation required here. To do so, we break the force on a particle into two parts: that due to particles in a region of size $\eta \ll \ell_{CG}$ about it, and the residual force from particles outside this region. The local part of the force will have a negligible effect on the motion of the center of mass of the regions of size $\ell_{CG} \gg \eta$, and so we can neglect it. The second piece of the force can be linearized in the relative displacements as in PLT, which now remains until particles move a distance of order η. Because the dynamical matrix has been modified only up to a finite scale, the diagonalisation which gives the eigenvectors and eigenvalues for the displacement field will converge to the same behavior at small k as in the full PLT. The ZA will therefore still be a valid approximation in this limit, with corrections at any finite k which can in principle be calculated. Thus, in summary, we expect the ZA to be a good approximation as required, if the scale of non-linearity is smaller than ℓ_{CG} at time t_{CG}[25]. We also expect, as in the previous case of CG at the initial time, that, at a given length scale, the deviations from ZA will diverge as the time for evolution with this "modified PLT" description does. We therefore expect the result to break down if the initial δ in the mother particle distribution (and therefore that in the CG distribution also, since they are equal) becomes itself arbitrarily small.

The CG at a finite time, defined as we have just discussed, is useful with respect to understanding the "self-similar" properties observed in the evolution of a SL, extensively discussed in [1] (and summarized in Sect. II C above). In the case that the CG is just a rescaling of the initial SL (i.e. in the approximation that the ZA is exact), it follows that

$$\xi_{CG}(r, t_{CG} + t) = \xi\left(\frac{r}{\alpha}, t\right) = \xi(re^{-2t_{CG}/5\tau_{\rm dyn}}, t) \quad (43)$$

where ξ_{CG} is the correlation function measured in the evolved CG system and $\xi(r)$ that in the original one. Now, if the CG system indeed reproduces the evolution of the original system, i.e., $\xi_{CG}(r,t) = \xi(r,t)$ for $t > t_{CG}$, the self-similarity relation Eq. (15) follows.

[25] This will necessarily be the case here as the δ of the original particle distribution, which controls the time at which any scale goes non-linear, has been chosen small compared to unity (but not too small, as discussed above).

B. Numerical simulations

Our original ("mother") system is the same SL64 initial condition considered in the previous section. Our coarse-grainings are now performed at the same length scales, i.e., $\alpha = 2$ and $\alpha = 4 = 2^2$. Following the prescription just given we therefore coarse-grain the original particle distribution evolved to times t_{CG} and $2t_{CG}$ respectively, where

$$t_{\rm CG} = \frac{5}{2}(\ln 2)\tau_{\rm dyn} \approx 1.73\tau_{\rm dyn}\,. \quad (44)$$

Unlike the case of the CG at $t = 0$, the Eulerian and Lagrangian CGs are in each case not the same. We thus define for each case two CGs so that we have four in total CG32$_t$, CG32$'_t$, CG16$_t$ and CG16$'_t$. The unprimed acronyms refer to the Eulerian (non-equal mass) simulations, the primed ones to the Lagrangian (equal mass) simulations. Since the IC are actually such that particles have moved only a small distance compared to ℓ_{CG} at the time of the corresponding coarse-graining, the differences should not be very important, as they come only from particles close to the surface of the CG cell being re-attributed to neighboring cells.

In Fig. 5 we give a schematic representation of this CG scheme. In Fig. 6 are shown snapshots in time of the evolved SL64, CG32$_0$ and CG32$'_t$ systems starting from a time $t = 2$ just a little after $t_{CG} \approx 1.73$ at which the latter distribution is defined. CG32$_0$ and CG32$'_t$ are clearly very similar, but visually one can discern that CG32$'_t$ traces slightly better than CG32$_0$ the morphology and clumpiness of the structures in the mother particle distribution.

In Fig. 7 is shown the evolution of the two-point correlation function in SL64, CG32$'_t$ and CG16$'_t$. We do not show here results for the Eulerian CGs because the (number-number) correlation function we can estimate straightforwardly is not equal in this case to the density-density correlation function (which is the quantity we are interested in). As in the previous section we see that the CG particle distributions begin to develop non-linear correlations later than in the mother distribution, but that the relative difference between the correlations in the two particle distributions becomes negligible subsequently. In Fig. 8 is shown the temporal evolution of the PS in the different particle distributions. In this case we show both the Lagrangian and Eulerian CG distributions as our estimator of the PS is indeed that of the mass density fluctuations [i.e. it takes into account the different masses of the particles, cf. Eq (7)]. We see again qualitatively very similar results to those in the previous section. The CGs, when they are initially defined, have

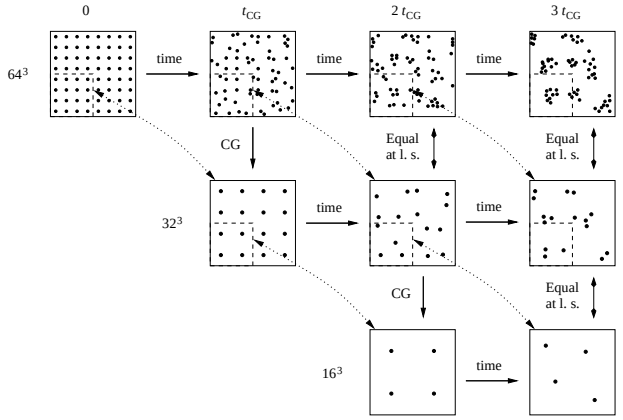

FIG. 5: Schematic representation of the "CG in time" experiment. The 64^3, 32^3 and 16^3 systems are represented from top to bottom, and the time evolution is represented in steps of $t_{\rm CG}$ from left to right. The arrows marked "CG" indicate that a CG is done (note that in the experiment done, the CG for the 16^3 system is obtained from the 64^3 system and not from the 32^3). The double arrows marked "l. s." *(large scales)* indicate that the system should share the same fluctuations at sufficiently large scales. The dashed arrows and the dashed squares illustrate the "self-similarity" argument: a square defined (partially or wholly) by the dashed lines are supposed to be identical up to a rescaling of length scales to the system to which the dashed arrow points.

small correlations which are indistinguishable in the figures from those in the mother particle distribution, but at larger k (of order the inverse of the CG scale ℓ_{CG}) they differ. We see also the small differences at these scales between the Eulerian and Lagrangian CG. As in the previous section we see that the relative differences between the evolved particle distribution decreases in time, with a greater convergence as non-linearity develops at smaller scales. Each PS must converge to the corresponding Poisson noise level at large k, which is simply the inverse of the particle density. Outside this range the PS of each CG interpolates rapidly, in the evolved particle distributions, to the PS of the mother distribution.

As discussed above, the CG at a finite time we have defined is useful for understanding the self-similar properties of the asymptotic evolution from SL initial conditions[1]. Indeed it has the property that, if the ZA is exact, the CG particle distribution defined in this way is identical (statistically) to the initial condition up to a rescaling of length scales. The "self-similarity" is then a direct consequence of the fact that the evolution before the onset of non-linearity on a given scale depends to a very good approximation only on the initial fluctuations at these scales, which themselves have (by construction) no intrinsic length scale, and the purely fluid evolution of these fluctuations described by the ZA.

In this respect it is interesting to check directly to what extent the CG mass distribution actually is a rescaling of the mother distribution, i.e. to what extent the ZA is an accurate approximation to the motion of the center of masses at the scale ℓ_{CG}, by direct comparison of the particle distributions. Further it is interesting to see how well the relation Eq. (43) is obeyed. In Figs. 9 and 10 we show these comparisons. In the first the dimensionless combination $n_0P(k)$ is plotted as a function of the k wavenumber, rescaled as required to the Nyquist frequency in each particle distribution (this corresponds to a rescaling of length scales by α). In Fig. 10 likewise the x axis has been rescaled in the same manner [26]. We see that the particle distributions are indeed initially very similar — except for some visible differences close to the Nyquist frequency, due to deviations from the ZA at these small scales — and in time become more similar as the evolution develops.

VI. CONCLUSIONS AND DISCUSSION

We have studied the evolution of an infinite SL of points under its self-gravity by comparing it with that of its particle CGs, using two different prescriptions for the latter. The primary aim is to get insight into the dynamics of the non-linear gravitational clustering in this system, which has been shown in [1] to be very similar qualitatively to that in simulations of structure formation in cosmology. We first summarize the main conclusions we draw from our study. We then discuss one specific question, which is important with respect to the theoretical interpretation of these results: the definition of a limit in which a continuum theoretical description should be appropriate to fully understand the evolution (in the linear *and* non-linear regime) of this system. This question is of particular relevance in the context of qualitatively similar simulations in cosmology, as such simulations are useful insofar as they reproduce the evolution of such a limit. How they do so is, however,currently poorly understood. Finally we briefly mention some directions for further work.

A. Summary of results

Our principal results and conclusions are the following:

- In the *numerical simulations* the coarse-grained particle distributions are observed to evolve to give, after a sufficient time, two-point correlation properties which agree well, over the range of scales simulated, with those in the original distribution. Indeed both the original system and its coarse-grainings converge toward a simple dynamical scaling ("self-similar") behavior *with the same amplitude*. The characteristic time required for the CG

[26] See the footnote in discussion of Fig. 3 above for the explanation of why not all the predicted Bragg peaks are visible in the plot.

FIG. 6: Snapshots of the evolution of SL64 (left column), $CG32_0$ and $CG32'_t$ (right column) at times 2,4,6 and 8. The points shown are obtained by projection on the x–y plane of a slice of thickness 0.4 along the z axis.

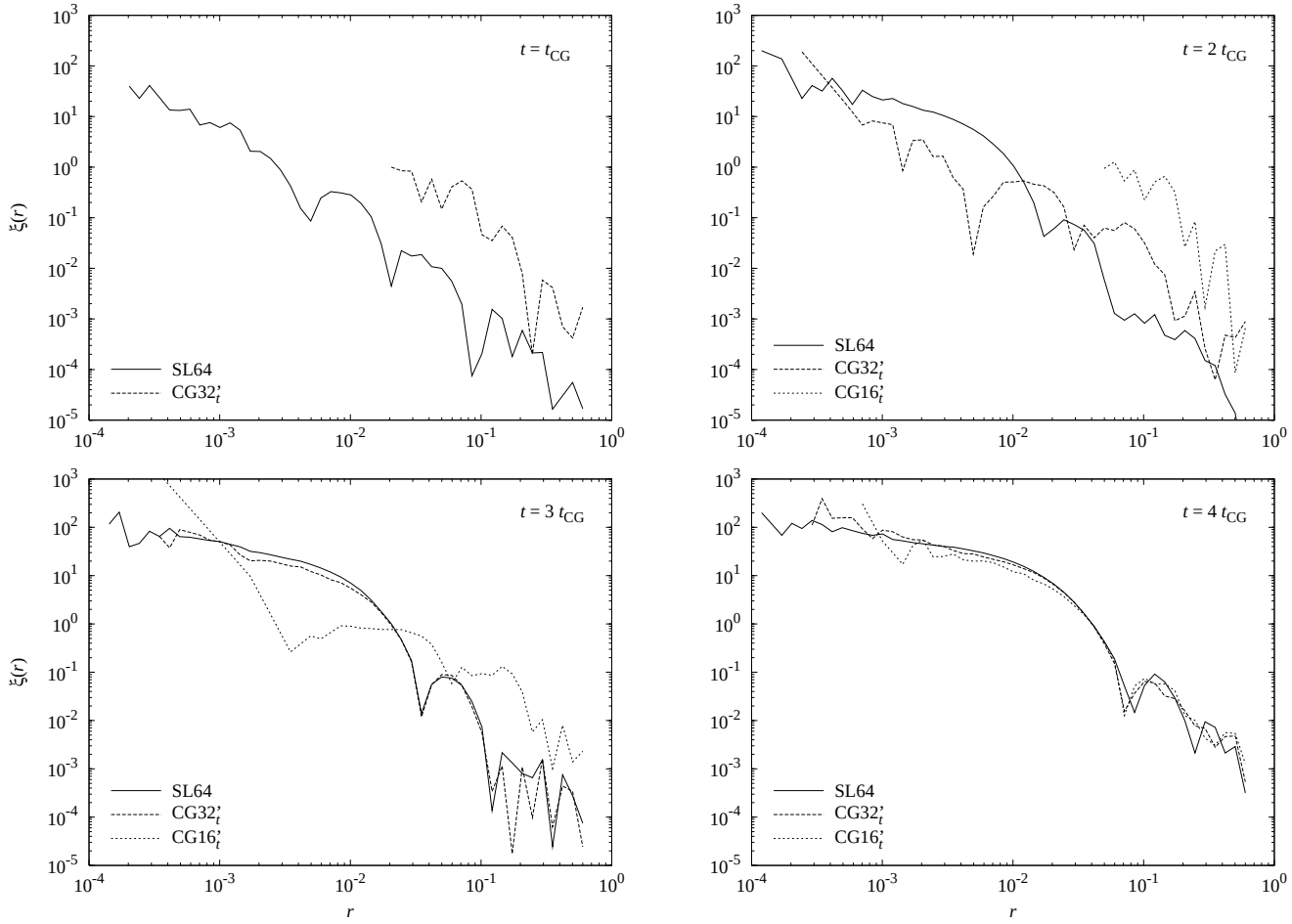

FIG. 7: Evolution of $\xi(r)$ in SL64 and the Lagrangian (equal mass) coarse-grainings $CG32'_t$ and $CG16'_t$. The different plots correspond to the times 1, 2, 3 and 4 in units of t_{CG}.

system to begin to reproduce the clustering in the original particle distribution at scales *below* the CG scale increases as the latter scale does. As discussed in Sec. IV A these observations are all very much in line qualitatively, and to a first approximation (see below) quantitatively, with the qualitative picture of the evolution of clustering widely accepted in cosmology: the CG distributions share the same fluctuations at large scales and it is these initial fluctuations alone, to a very good approximation, which determine the correlations which develop at smaller scales at later times.

- Once particles begin to fall on one another in any of the SL simulations, there is a phase in which very significant non-linear correlations develop due to interactions between NN pairs of particles. Further we have seen, as in [1, 15], that the *form* of the two-point correlation function which develops in this phase is very similar to that observed, in the same range of amplitude, in the asymptotic scaling regime at later times. Thus it appears that it is always possible to choose a CG of the original system, which reproduces quite well the non-linear correlations in the original system with this "early time", explicitly discrete, dynamics of "macro-particles" of the CG distribution.

  This provides a simple physical picture/dynamical model for the generation of the non-linear correlation function in the relevant range (up to an amplitude of ξ in the range of $10 - 10^2$). We will describe more quantitatively in a forthcoming paper [30] the exact form of the non-linear correlation function given by this model.

  This finding is *prima facie* very different, in spirit and substance, to any existing explanations of the dynamics giving rise to non-linear correlations in

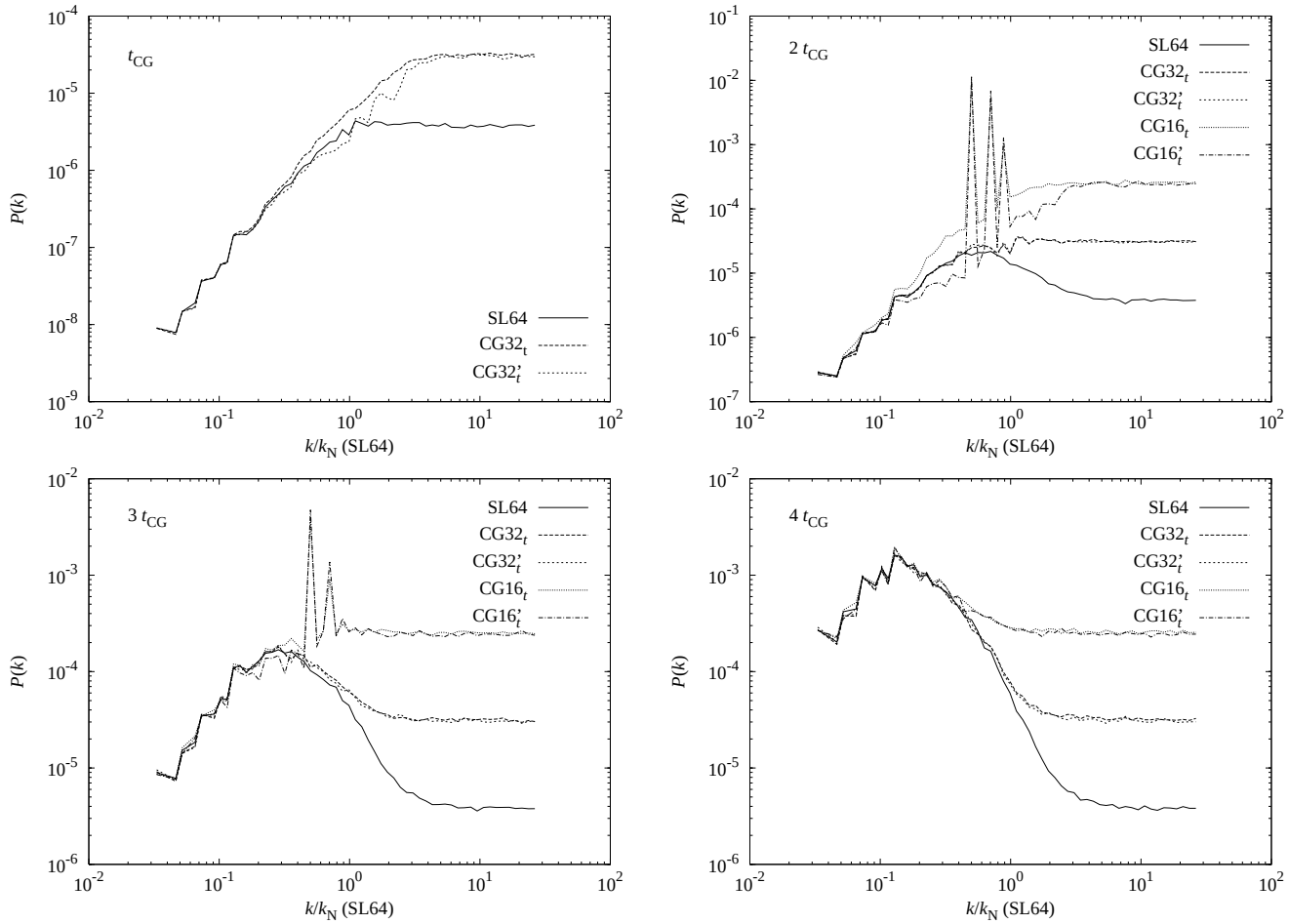

FIG. 8: Evolution of the PS in SL64 and the different CG in time (see the legend). The different plots correspond to the times 1, 2, 3 and 4 in units of $t_{\rm CG}$.

N body simulations in cosmology. In this context theoretical modeling invariably assumes that the non-linear correlations observed in simulations in this range should be understood in the framework of a continuum Vlasov limit, in which a mean-field approximation of the gravitational field is appropriate. Indeed the fact that self-similarity is observed, with a behavior independent of the particle density, is usually taken as an indication that such a continuum description is appropriate. Our model is manifestly not of this type (while also consistent with the amplitudes of the correlation function being independent of particle density). There is, of course, not necessarily a contradiction: it may be that our simple discrete model is simply a sufficiently good approximation to the continuum model[27]. We will return to a discussion of this point below.

- A closer examination of the results shows that there are small, but observable, differences in amplitudes between the original SL and it CGs. These may be explained using a more accurate description of the early time evolution of the system, given by the perturbative PLT treatment of [11, 12], valid up to the onset of the NN interaction phase. In this framework the fluid limit criterion Eq. (39) for the end of this phase may be replaced by a more accurate criterion, taking into account the dependence of the rate of evolution of modes of the displacement field at small scales on particle density (and their wavelength). Generically the effect of discreteness (i.e. the difference with respect to the fluid limit) is to slow down the growth of modes

[27] In [31] we have presented an analysis of the force distribution in cosmological simulations at the time of formation of the first non-linear correlations, showing that NN interactions also dominate in this case. Thus we do not expect that either the different initial conditions, or the fact that there is expansion of the spatial background, should qualitatively change the early time dynamics with respect to that of the system discussed here.

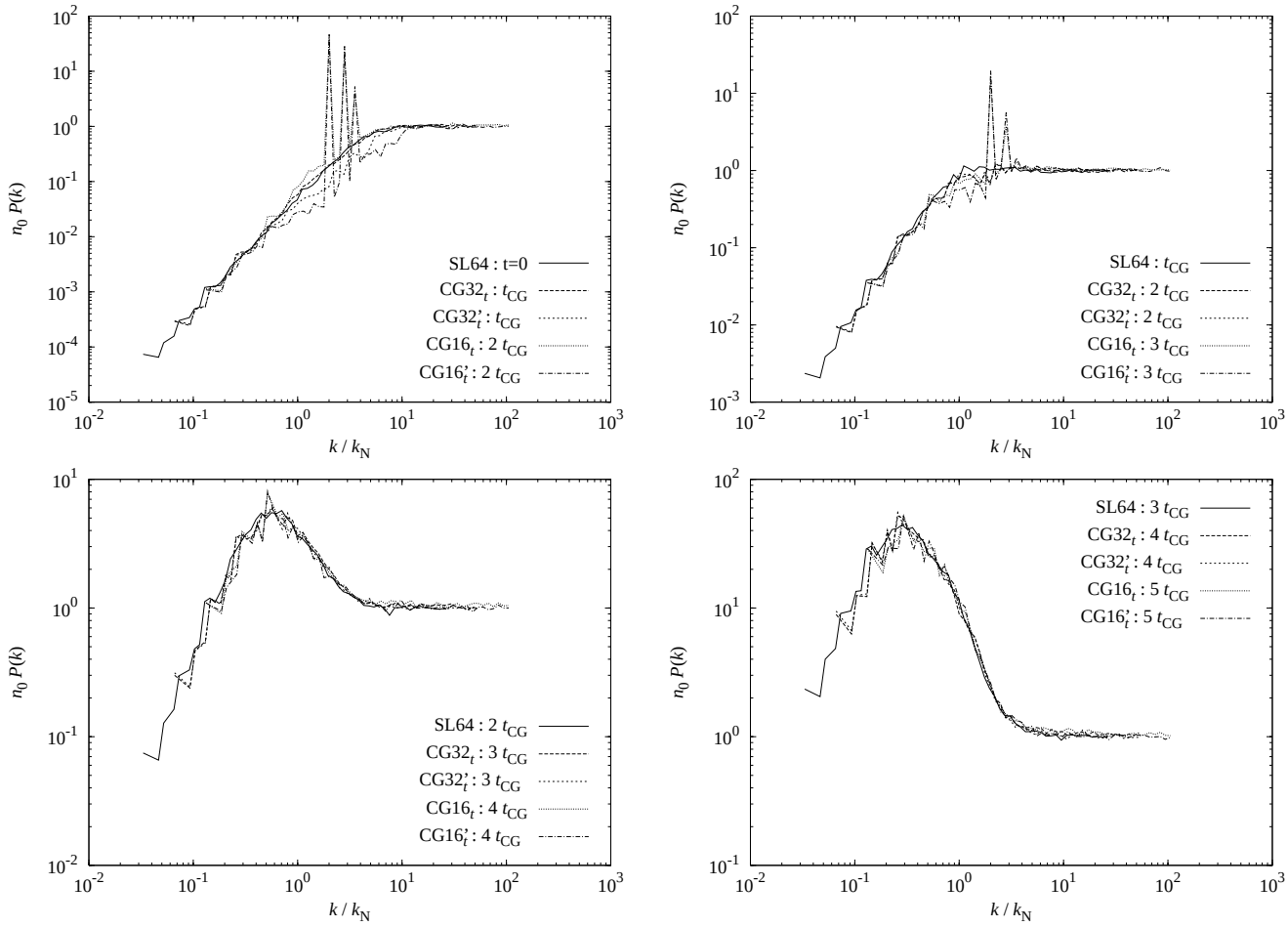

FIG. 9: Comparison of the PS in SL64 at times $0, 1, 2, 3$ in units of $t_{\rm CG}$ with the PS in CG32$'_t$ at times $1, 2, 3, 4$ and the PS in CG16$'_t$ at times $2, 3, 4, 5$. We have plotted $n_0 P(k)$ as a function of k/k_N, where k_N is the Nyquist frequency for the given distribution. In the approximation that the ZA is valid at the scale of coarse-graining, all the plots should overlap.

at smaller scales (comparable to the inverse of the inter-particle spacing). Thus the time of exit from the PLT phase is longer than given by Eq. (39). In our numerical simulations a resultant "lagging" of the non-linear correlation function in the CG simulation is just discernible at early times. These effects are small, and visible only at very early times, because the value of δ used, albeit small compared to unity, is not so small. In the limit that the normalized shuffling δ becomes arbitrarily small, however, these differences between the discrete and fluid system can become arbitrarily large.

- Since the coarse-grainings defined on the initial conditions give $\delta \propto (\ell/\ell_{CG})^{5/2}$, differences between the full PLT evolution and the fluid evolution will, for the reasons explained in the previous paragraph, diverge in the limit $(\ell/\ell_{CG}) \to 0$. The coarse-grainings defined at finite times avoids this feature by construction: it ensures that the CG system has approximately the same δ, so that the time spent in the phase in which discreteness effects accumulate as described by PLT is identical to that in the mother particle distribution. Thus the use of a CG distribution, evolving in its "early" phase, to reproduce the non-linear correlations in the original distribution would be expected to work in this case even for arbitrarily large times (which require arbitrarily large values of ℓ_{CG}).

- The origin of self-similarity observed in the evolution of the system, in which the temporal evolution of the correlation function is equivalent to a rescaling of spatial variables, is nicely illustrated using the CG at finite times. Indeed insofar as the system, for its further evolution, can be approximated by its CG defined in this way, one obtains self-similarity. We have seen that there are thus two ingredients giving this behavior. Firstly the non-linear correlations which develop at a given scale depend on the initial conditions only through the fluctuations present initially at larger scales (i.e. the same as the first point above). Secondly these larger scales are well described at early times by the

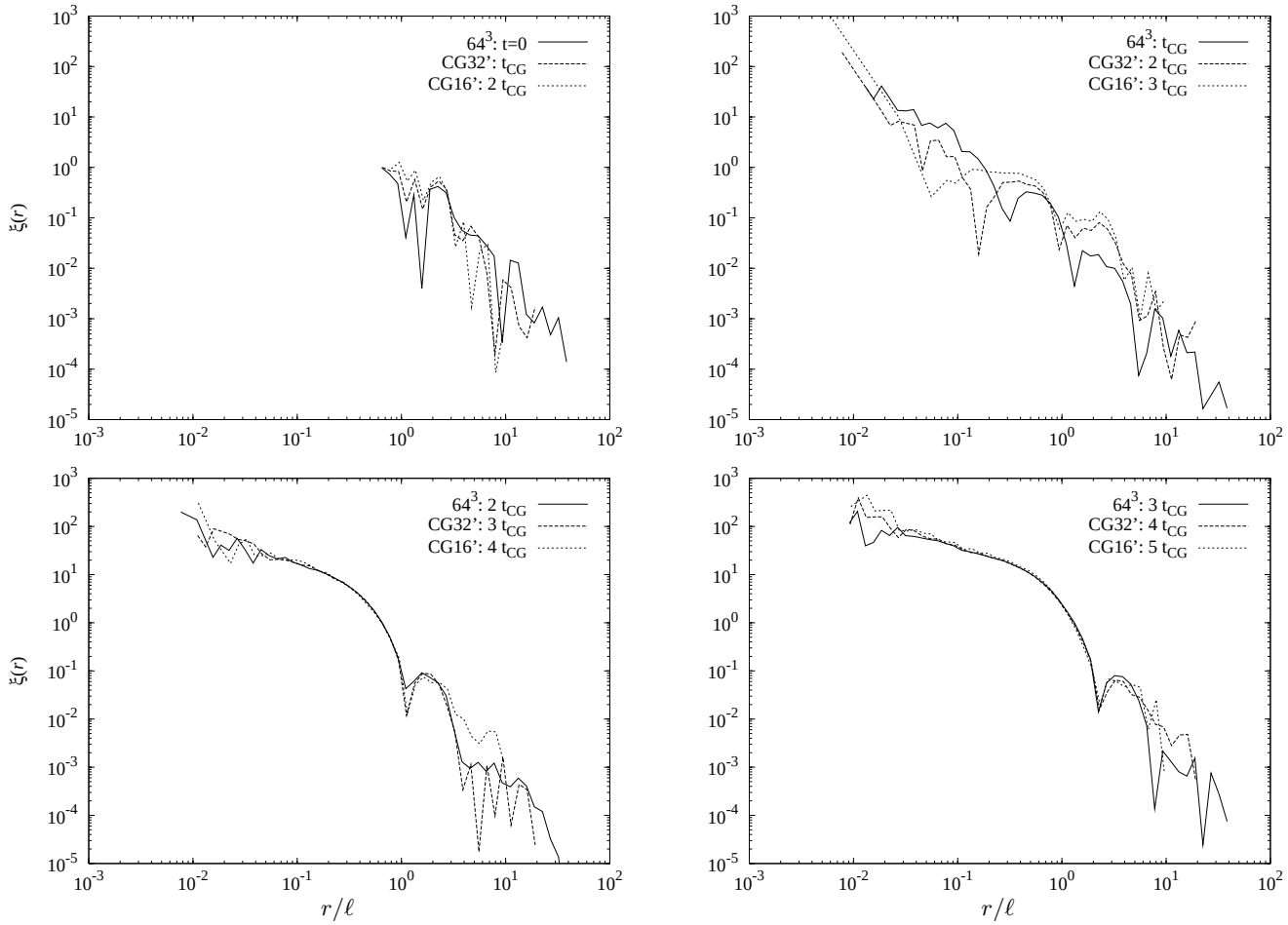

FIG. 10: Comparison of $\xi(r)$ in SL64 at times $0, 1, 2, 3$ in units of $t_{\rm CG}$ with $\xi(r)$ in CG32$'_t$ at times $1, 2, 3, 4$ and in CG16$'_t$ at times $2, 3, 4, 5$. As in the previous figure, the approriate rescaling of lengths on the x-axis has been done so that these functions should overlap in the approximation that ZA is exact at (and above) the CG scale.

Zeldovich approximation, i.e., by the purely fluid limit in which the evolution is independent of the particle density.

B. The continuum limit

As noted in the introduction, one of the motivations for studying coarse-grainings like those we have considered here is that such a study may help to clarify what properties of the evolved SL may be understood as valid in a continuum limit: the idea is that such properties should be invariant under coarse-graining, i.e., invariant with respect to changes in the small scale discrete structure of the systems. The fact that we have found that the evolution observed in the original system is not completely invariant in this sense, and indeed that the different coarse-graining procedures give different results, shows that this evolution is not fully representative of such a continuum limit. The question of such a limit is of relevance if one wants to determine whether, in particular, the *form* of the correlation function in the self-similar should be derivable with such a continuum theory. As mentioned above, this is particularly important to understand in the context of simulations of this type in cosmology, as the goal of the N body method is to reproduce such a limit.

To address this question we need to make more precise what we mean by "continuum limit". There is in fact, of course, no unique way of defining such a limit. Implicitly above we have been supposing a limit in which the inter-particle spacing ℓ goes to zero in some appropriate manner, so that a hydrodynamic type description of the system becomes valid. More specifically, in cosmology and more generally for systems with long range interactions, one is interested in identifying the Vlasov limit, in which the system is described by a one particle phase space density obeying a Liouville equation in which the force is calculated in a self-consistent mean field approximation (see, e.g. [18, 32]).

The SL system we have studied is characterized [1] by a single *dimensionless* parameter: the normalized shuf-

fling parameter δ. Indeed the evolution we have studied is, as discussed in [1], independent of both the system size and force softening. Two SL with the same δ, but different ℓ, are then equivalent because of the scale free nature of gravity. To give meaning to a limit $\ell \to 0$ we must, therefore, evidently introduce at least one additional length scale. We note that in the discussion of the ZA above, in Sec. II E, we have derived it as a continuum limit by fixing the wave length of the mode considered and taking $\ell \to 0$. In this way we established that long wavelength modes of the displacement field behave as derived from a set of fluid equations for the self-gravitating fluid. The SL system *as a whole*, however, cannot have any continuum limit in the desired sense unless we introduce an additional length scale in the system.

In [33] it is shown formally that such a Vlasov description is recovered in the limit that $N \to \infty$ for a class of long-range interacting system of N particles, provided their interaction potentials are regulated at small separations. This suggests that one way in which we should recover such a continuum description is by introducing such a regularization, characterized by a scale ε. Then one can take $\ell \to 0$ at fixed ε (and then, in principle $\varepsilon \to 0$).

Following what was said above, we expect that in this limit it should be possible to define coarse-graining procedures like we have done which leave the evolution of the system invariant. It is not difficult to see that this should indeed be the case. Let us assume that the force falls rapidly to zero at separations less than ε. The PLT treatment can be applied and will give a spectrum of modes which are modified with respect to full gravity for $k\varepsilon > 1$. In the limit $\ell/\varepsilon \to 0$ these modifications will become independent of ℓ. Thus the evolution of the displacement field will be independent of ℓ. As the CG defined are just a modification of this latter scale, they will have no effect on the evolution (provided $\ell_{CG}/\varepsilon \ll 1$). In practice this means that we can coarse-grain the system of finite ℓ (and $\ell \ll \varepsilon$) up to any ℓ_{CG}, provided $\ell_{CG} \ll \varepsilon$.

An alternative way of defining the limit $\ell \to 0$ is by introducing a characteristic scale not in the force, but in the fluctuations in the system. The SL has no such scale, but it is easy to introduce one by defining instead the initial displacement field so that it has such a cut-off: we derive it as the sum of modes in reciprocal space, with all modes $k > k_c$ set equal to zero. We can then define a continuum limit for the system as $k_c\ell \to 0$. It is not difficult to see that a similar reasoning to that used for the previous case leads to the conclusion that coarse-graining like those we have defined now change negligibly the evolution of the system provided $k_c\ell_{CG} \ll 1$.

We note that there is one intrinsic problem with the latter prescription: it works because the scale k_c is the one characterizing the shortest wavelength fluctuation in the system. This is only true, however, at the initial time, as gravity will progressively (and very efficiently) create fluctuations at smaller scales. Thus the continuum limit will only be valid for some finite time. For the former case this will not be the case as the finite value of ε will place a time-independent lower bound on the scales at which fluctuations develop. There is thus an intrinsic time independent separation between the scales of inhomogeneity and the particle scale.

This discussion tells us then how we should extrapolate our simulations toward a well defined continuum (Vlasov) theory to determine which results are valid in this limit: we should increase particle density until $\ell \ll \varepsilon$. Manifestly this will "kill" the discrete particle dynamics which we have used in our model to explain the *form* of the non-linear two-point correlation function. If this form is indeed that of the continuum limit it should be observed in simulations with $\ell \ll \varepsilon$ when non-linear correlations develop for $r \gg \varepsilon$. To perform simulations to test whether this is so requires that there be a reasonable range of scale $r \gg \epsilon \gg \ell$ well inside the simulation box, which implies a particle number much greater than that in the largest 64^3 simulation discussed here (Taking, e.g., $\ell = 0.2\varepsilon$, ε is already almost one tenth of the box size). With the particle numbers, in the range $10^8 - 10^{10}$, now being accessed by several gravitational N body simulation groups (see, e.g., [34, 35, 36]) such numerical tests are, however, in principle feasible and would yield clear conclusions on this very basic question.

Acknowledgments

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